

Causality in spin foam models for quantum gravity¹

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Abstract. We describe how the Barrett-Crane spin foam model defines transition amplitudes for quantum gravity states and how causality can be consistently implemented in it.

1. Introduction

Spin foam models [2][3] have emerged recently as a new promising approach to the construction of a quantum theory of gravity, being an explicit implementation of the path integral approach to quantum gravity, where amplitudes for gravity states are defined as a sum over all the 4-geometries interpolating between given boundary 3-geometries, weighted by the action for gravity, with an additional sum over 4-manifolds:

$$Z(h_1, h_2) = \sum_{\mathcal{M}} \int_{h_1, h_2} \mathcal{D}g e^{i S_{gr}}. \quad (1)$$

Spin foam models are constructed out of only combinatorial and algebraic data (coming from the representation theory of the Lorentz group) and continuum geometric structures are expected to emerge as an approximation in some appropriate limit. Causality is of course a crucial ingredient for this to happen, since a classical metric is determined almost completely by the causal structure of spacetime, with the remaining degree of freedom being just a length scale. It is also a crucial element to understand what kind of transition amplitudes spin foam models define. In fact, many different amplitudes may be given by a path integral realization even for the simple case of a relativistic particle in flat space, with action $S(x) = \int_{\lambda_1}^{\lambda_2} (p_\mu \dot{x}^\mu - T \mathcal{H}) d\lambda$, where $\mathcal{H} = p_\mu p^\mu + m^2 = 0$ is the Hamiltonian constraint and T is the proper time elapsed between the initial and final state. An integral over both $T < 0$ and $T > 0$ yields the Hadamard Green function:

$$\begin{aligned} G_H(x_1, x_2) &= \langle x_2 | x_1 \rangle = \int_{-\infty}^{+\infty} dT \int dx dp e^{i \int d\lambda (px - T\mathcal{H})} = \int dx dp \delta(p^2 + m^2) e^{i \int d\lambda x p} = \\ &= G^+(x_1, x_2) + G^-(x_1, x_2) = G^+(x_1, x_2) + G^+(x_2, x_1) = G_H(x_2, x_1), \end{aligned} \quad (2)$$

¹ This work is a much shortened presentation of the results published in [1], to which we refer for a more detailed account and a complete set of references.

where the G^\pm are the Wightman functions. G_H solves the Klein-Gordon equation in both its arguments, does not register any order between them, is an a-causal amplitude between physical states and defines a generalized projector operator from kinematical states onto solutions of the Hamiltonian constraint: $G_H(x_1, x_2) = \langle x_2 | x_1 \rangle_{phys=kin} \langle x_2 | \mathcal{P}_{\mathcal{H}=0} | x_1 \rangle_{kin}$. An integral over only $T > 0$, so that $|x_2\rangle$ lies always in the future of $|x_1\rangle$, gives the Feynman propagator or causal amplitude:

$$G_F(x_1, x_2) = \langle x_2 | x_1 \rangle_C = \theta(x_1^0 - x_2^0) G^+(x_1, x_2) + \theta(x_2^0 - x_1^0) G^-(x_1, x_2), \quad (3)$$

which is not a solution of the Klein-Gordon equation, does not realize the projection operator, but is a transition amplitude which takes into account causality. Therefore, after having recognised a spin foam model as a realization of the a path integral for quantum gravity, one has still answer several key questions: what kind of amplitude does it define? is it an implementation of the projector operator or a realization of the Feynman propagator? if it defines a projector, where is encoded the \mathbf{Z}_2 symmetry between T and $-T$? how to break such a symmetry and implement causality?

2. The (quantum) geometry of the Lorentzian Barrett-Crane model

The Barrett-Crane spin foam model[4][3][2] is a path integral quantization of the action:

$$S(\omega, B, \phi) = \int_{\mathcal{M}} \left[B^{IJ} \wedge F_{IJ}(A) - \frac{1}{2} \phi_{IJKL} B^{IJ} \wedge B^{KL} \right] \quad (4)$$

which is a BF theory with variables a 2-form $B_{\mu\nu}^{IJ}$ and a 1-form connection A_μ^{IJ} (with curvature $F_{\mu\nu}^{IJ}$), all with values in $so(3, 1)$, but with a constraint on the B field enforced by the Lagrange multiplier ϕ_{IJKL} . The constraints on the B field have four sectors of solutions[3]: $B^{IJ} = \pm e^I \wedge e^J$ and $B^{IJ} = \pm \frac{1}{2} \epsilon^{IJ}{}_{KLE} e^K \wedge e^L$, so in one of these sectors: $S \rightarrow S_{EH} = \int \epsilon_{IJKL} e^K \wedge e^L \wedge F^{IJ}$, i.e. the theory reduces to pure 1st order Einstein gravity. The other sector, differing by a global change of sign only, is classically equivalent to this, while the other two have no geometrical interpretation. The \mathbf{Z}_2 symmetry between the two geometric sectors directly affects the path integral quantization, since a change of sign in the B field is equivalent to a change of sign in the lapse function, and thus in the proper time. We replace the continuum manifold by a simplicial complex, or by its dual 2-complex, and the continuum fields by discrete variables. In particular, the B field is associated to the triangles of the triangulation by: $B^{IJ}(t) = \int_t B_{\mu\nu}^{IJ}(x) dx^\mu \wedge dx^\nu \in so(3, 1) \simeq \wedge^2(\mathbf{R}^{3,1})$, and the constraints on the B field become constraints on the bivectors B^{IJ} [4]. The quantization[4] then proceeds by associating to each triangle an irreducible representation of the Lorentz group in the principal unitary series (labelled by $n \in \mathbf{N}$ and $\rho \in \mathbf{R}_+$), with the identification: $B^{IJ}(t) \leftrightarrow *J^{IJ}((n, \rho)_t)$, where the J 's are the generators of the Lorentz algebra, and assigning to each tetrahedron a tensor in the tensor product of the four representation spaces of its triangles. The constraints on the bivectors then become constraints on the representations used and on these tensors. To obtain the Barrett-Crane partition function one may proceed in several ways [3][7][9]. In each case one starts from BF theory and imposes the BC constraints on this using suitable projectors. The resulting model, using only the simple representations $(0, \rho)$, is:

$$Z = \left(\prod_f \int_{\rho_f} d\rho_f \rho_f^2 \right) \left(\prod_{v, e_v} \int_{H^+} dx_{e_v} \right) \prod_e \mathcal{A}_e(\rho_k) \prod_v \mathcal{A}_v(\rho_k, x_i) \quad (5)$$

with the amplitudes for vertices (4-simplices) being given by:

$$\begin{aligned} \mathcal{A}_v(\rho_k, x_i) = & K^{\rho_1}(x_1, x_2) K^{\rho_2}(x_2, x_3) K^{\rho_3}(x_3, x_4) K^{\rho_4}(x_4, x_5) K^{\rho_5}(x_1, x_5) \\ & K^{\rho_6}(x_1, x_4) K^{\rho_7}(x_1, x_3) K^{\rho_8}(x_3, x_5) K^{\rho_9}(x_2, x_4) K^{\rho_{10}}(x_2, x_5), \end{aligned} \quad (6)$$

and those for edges (tetrahedra) being also explicitly known and considered part of the measure[1]. The functions K (one for each face (triangle)) have the explicit expression: $K^{\rho_k}(x_i, x_j) = \frac{2 \sin(\eta_{ij} \rho_k / 2)}{\rho_k \sinh \eta_{ij}}$, where $\eta_{ij} = \cosh^{-1}(x_i \cdot x_j)$ is the hyperbolic distance between the points x_i and x_j on the hyperboloid $H^+ = \{x^\mu \in \mathbf{R}^{3,1} / x \cdot x = 1, x^0 > 0\}$. The partition function above should be understood as just a term within a sum over triangulations or over 2-complexes, to restore the full dynamical content of the quantum gravity theory. Upon quantization, the triangle areas are given by $A_t^2 = B_t^{IJ} B_{tIJ} = -J^{IJ}(\rho_t) J_{IJ}(\rho_t) = \rho_t^2 + 1 > 0$, so the ρ 's determine the areas of the triangles, and we see that all the triangles are spacelike (consequently, also all the tetrahedra are spacelike). The same result can also be confirmed by a canonical analysis of the area operator[10]. The $x \in H^+$ variables are un-oriented normals to the tetrahedra of the manifold, the oriented normals being $n_i = \alpha_i x_i$, with $\alpha = \pm 1$ for future- and $\alpha = -1$ for past-oriented n . The Barrett-Crane model corresponds to a piecewise flat manifold, with patches of flat space-time, the 4-simplices, glued together along their common tetrahedra. To each 4-simplex is attached a local reference frame, and there is a non-trivial connection rotating from one to another. The Lorentz invariance at each 4-simplex may be used to fix one of the vectors x , so the true variables are the hyperbolic distances η between any two of them. These in turn correspond to the dihedral angles, $\theta_{ij}^\sigma = \alpha_i \alpha_j \eta_{ij}$, between two tetrahedra sharing a triangle. Therefore the underlying classical theory for the Barrett-Crane model, somehow hidden in the above formulation, is a first order formulation of Regge Calculus based on angles and areas as fundamental variables[1], with the angles constrained by the Schläfli identity: $\sum_{i \neq j} A_{ij} d\theta_{ij} = 0$. What are the quantum gravity states in the Barrett-Crane model? Let us consider a spin foam with boundary, this being made of 4-valent vertices glued to form an oriented graph. The boundary states are Lorentz invariant functionals of group elements g_e living on the edges of the graph and normals x_v at each vertex, invariant with respect to $SU(2)$ at each vertex and edge (simplicity constraints). This space of functionals is endowed with the $SL(2, \mathbf{C})$ Haar measure, and an orthonormal basis for the resulting Hilbert space of L^2 functions is given by the simple spin networks:

$$s^{\{\rho_e\}}(g_e, x_v) = \prod_e K_{\rho_e}(x_{s(e)}, g_e \cdot x_{t(e)}) = \prod_e \langle \rho_e x_{s(e)}(j=0) | g_e | \rho_e x_{t(e)}(j=0) \rangle, \quad (7)$$

where $|\rho x(j=0)\rangle$ is the vector of the ρ representation invariant under the $SU(2)$ subgroup leaving the vector x invariant. This same Hilbert space for kinematical states comes out of a canonical analysis in an explicitly covariant framework[10]. Now the main issue is: what kind of transition amplitudes are those defined by the Barrett-Crane model? We argue that it is a realization of the projector operator for quantum gravity[1], as argued also in [6][8]. The task is to locate clearly where in the model the \mathbf{Z}_2 symmetry relating positive and negative proper times is implemented. As we said, this symmetry originates from the symmetry $B \rightarrow -B$ in the bivector field, and in our discretized context corresponds to a change in the orientation of the triangles in the simplicial manifold, and consequently of the tetrahedra. Let us use the unique decomposition of $SL(2, \mathbf{C})$ representation functions of the 1st kind K into representation functions of the

2nd kind K^\pm and write the K functions as

$$\begin{aligned} K^{\rho_{ij}}(x_i, x_j) &= \frac{2 \sin(\eta_{ij} \rho_{ij}/2)}{\rho_{ij} \sinh \eta_{ij}} = \frac{e^{i \eta_{ij} \rho_{ij}/2}}{i \rho_{ij} \sinh \eta_{ij}} - \frac{e^{-i \eta_{ij} \rho_{ij}/2}}{i \rho_{ij} \sinh \eta_{ij}} = K_+^{\rho_{ij}}(x_i, x_j) + K_-^{\rho_{ij}}(x_i, x_j) = \\ &= K_+^{\rho_{ij}}(x_i, x_j) + K_+^{\rho_{ij}}(-x_i, -x_j) = K_+^{\rho_{ij}}(x_i, x_j) + K_+^{\rho_{ij}}(x_j, x_i) = K_+^{\rho_{ij}}(\eta_{ij}) + K_+^{\rho_{ij}}(-\eta_{ij}) \end{aligned} \quad (8)$$

and make the following alternative expressions of the same Z_2 symmetry manifest:

$$K^{\rho_{ij}}(x_i, x_j) = K^{\rho_{ij}}(\eta_{ij}) = K^{\rho_{ij}}(-\eta_{ij}) = K^{\rho_{ij}}(-x_i, -x_j) = K^{\rho_{ij}}(x_j, x_i) = K^{-\rho_{ij}}(x_i, x_j). \quad (9)$$

We see that the symmetry characterizing the projector operator is indeed implemented as a symmetry 1) under the exchange of the arguments of the K functions, so the model does not register any ordering among the tetrahedra in each 4-simplex, 2) under the change of orientation of the two tetrahedra sharing the triangle, so it does not register its orientation either, 3) under a change in sign of the distance between the two points on H^+ , i.e. in the way the model uses upper and lower hyperboloids ($\eta \in H^+ \leftrightarrow -\eta \in H^-$), 4) under the exchange of a representation $(0, \rho)$ and its dual $(0, -\rho)$.

3. Implementing causality: a causal transition amplitude

We turn now to the problem of implementing causality in the Barrett-Crane model, i.e. to break consistently the identified Z_2 symmetry. This consistent restriction is found by requiring that the resulting amplitude has stationary points corresponding to good simplicial Lorentzian geometries[1]. For a given triangulation Δ , the amplitude reads

$$A(\Delta) = \sum_{\epsilon_t = \pm 1} \int \prod_t \rho_t^2 d\rho_t \prod_T A_e(\{\rho_t, t \in T\}) \prod_s \int_{(H^+)^4} \prod_{T \in s} dx_T^{(s)} \left(\prod_{t \in s} \frac{\epsilon_t}{i \rho_t \sinh \eta_t} \right) e^{i \sum_{t \in s} \epsilon_t \rho_t \eta_t}. \quad (10)$$

The action for a single (decoupled) 4-simplex is then

$$S = \sum_{t \in s} \epsilon_t \rho_t \eta_t = \sum_{t=(ij) \in s} \epsilon_{ij} \alpha_i \alpha_j \rho_{ij} \theta_{ij}, \quad (11)$$

with the angles θ_t constrained by the Schläfli identity, that can be enforced by a Lagrange multiplier $\mu \in \mathbf{R}$, and its stationary points are defined by

$$\epsilon_{ij} \alpha_i \alpha_j = \text{sign}(\mu) \quad \rho_{ij} = |\mu| A_{ij}. \quad (12)$$

This means that the area of the triangles are given (up to scale) by the representation labels ρ_{ij} and that we have a consistency relation between the orientation of the tetrahedra α_i , the orientation of the triangles ϵ_{ij} and the global orientation of the 4-simplex $\text{sign}(\mu)$. Now we can extend this orientation to the whole spin foam, imposing that if a tetrahedron is past-oriented for one 4-simplex, then it ought to be future-oriented for the other sharing it. A consistent orientation is thus a choice of μ_v and $\alpha_{T,v}$ (for each tetrahedron T attached to a 4-simplex v) such that: $\forall T, \mu_{v_1} \alpha_{T,v_1} = -\mu_{v_2} \alpha_{T,v_2}$ where v_1 and v_2 are the two 4-simplices sharing T . This is also equivalent to requiring an oriented dual 2-complex. Now we can fix the variables ϵ_t , and thus break the Z_2 symmetry that erases causality from the model (average over all possible orientations of the triangles), to the values corresponding to the stationary points in the a-causal amplitude (of course we do

not impose the equations of motion for A_t and θ_t). This leads to a *causal amplitude*[1], constructed by picking from 10 only the terms corresponding to the chosen ϵ_t :

$$A_{causal}(\Delta) = \prod_s \int_{(\mathcal{H}_+)^4} \prod_{T \in s} dx_T^{(s)} \prod_{t \in s} \frac{\epsilon_t}{i \rho_t \sinh \eta_t} \int \prod_t \rho_t^2 d\rho_t \prod_T A_e^T(\{\rho_t\}_{t \in T}) e^{i \sum_t \rho_t \sum_{s|t \in s} \theta_t(s)}, \quad (13)$$

with suitable boundary terms, to be understood within a sum over oriented 2-complexes or triangulations, provided with a consistent causal structure $(\Delta, \{\mu_v, \alpha_T, \epsilon_t\})$. The causal partition function is then basically of the form:

$$Z_{causal} = \sum_{\Delta} \lambda(\Delta) \int \mathcal{D}\theta_t(\Delta) \int \mathcal{D}A_t(\Delta) e^{i S_R^{\Delta}(A_t, \theta_t)} \quad (14)$$

What are its features? 1) It is a simplicial realization of the sum-over-geometries approach to quantum gravity, for a first order Lorentzian Regge action, with areas and dihedral angles being independent variables, and with a precise assignment of a measure and an additional sum over causally well-behaved triangulations, where the causal relations are encoded in the orientation; 2) the geometric variables have a natural algebraic characterization in the representation theory of the Lorentz group, and the combinatorial data used are from the 2-complex dual to the triangulation only, so the model is a spin foam model; 3) it realises the general definition given for a causal spin foam model (it is, to the best of our knowledge, its first non-trivial example), except for the use of the full Lorentz group instead of its $SU(2)$ subgroup; 4) it identifies causal sets (given by the “1st layer” of the dual 2-complexes) as the fundamental discrete structures on which quantum gravity has to be based, as in [5], but it contains additional metric data intended to determine a consistent length scale, which, in the traditional causal set approach, is meant to be obtained by “counting only”; this is a particular case included in the model and obtained by fixing all the geometric data to some arbitrary value; 5) also, if we fix all the geometric data to be those obtained from a fixed edge length, then what we obtain is the conventional sum over triangulations in the dynamical triangulations approach to quantum gravity, for Lorentzian triangulations; the additional integrals, if restored, can be interpreted as providing a sum over proper times, usually not implemented in that approach. In the new formulation presented above, the Barrett-Crane model fits [1] into the general scheme of quantum causal sets (or quantum causal histories)[11], being its first explicit non-trivial example. Consider an oriented graph, restricted to be 5-valent and to not include closed cycles of arrows, identified with the first *stratum* of the spin foam 2-complex. Interpreting the arrows as representing causal relations, this is a causal set, the orientation of the links reflecting the ordering relation among the vertices. Because of the restriction on the valence, it can be decomposed into building blocks, since for each vertex only one out of four possibilities may be realised, corresponding to the 4 possible Pachner moves ($4 - 1$, $3 - 2$, and their reciprocal), in the dual simplicial interpretation, giving the evolution of a 3-dimensional simplicial manifold in time. The crucial point is the identification of the direction of the arrow in the causal set with the orientation of the tetrahedron it refers to. The quantization is then the assignment of the Hilbert spaces of intertwiners among four given simple continuous representations of the Lorentz group, representing the possible states of the tetrahedra in the manifold, to the *arrows* of the causal set and of the causal Barrett-Crane amplitude defined above to its nodes, as evolution operator. Hilbert spaces that are a-causal to each other can be tensored together, so, in particular, for two a-causally related tetrahedra we can tensor the

corresponding intertwiners, so obtaining open spin networks with more than one vertex for each a-causal set of events. In each of the building blocks both the source and target arrows form an a-causal set[11], and together form a complete pair, so they are in turn linked by a causal relation in the poset of a-causal sets defined on the edge-poset[11]; to each causal relation among complete pairs, i.e. to each of the building blocks of the edge-poset, we associate the causal Barrett-Crane amplitude for a single 4-simplex. Also operators referring to a-causal sets that are not causally related to each other can be tensored together, so composite states constructed evolve according to composite evolution operators built up from the fundamental ones. We have already stressed from the beginning that a sum over 2-complexes is necessary to restore the full dynamical content of the gravitational theory. In this causal set picture this means that we have to construct a sum over causal sets interpolating between given boundary a-causal sets α and β , each poset weighted by the causal Barrett-Crane amplitude, so that the full evolution operator is $\mathcal{E}_{\alpha\beta} = \sum_c \lambda_c A_{\alpha\beta}^c$. The properties of $\mathcal{E}_{\alpha\beta}$ and $A_{\alpha\beta}^c$ (antisymmetry, reflexivity, transitivity, unitarity) are discussed in [1].

4. Conclusions

We described briefly the Lorentzian Barrett-Crane model, which is the most studied spin foam model for 4-d quantum gravity, and the classical and quantum description of spacetime geometry behind its formulation[1]. We explained why it has to be considered as providing the physical inner product between quantum gravity states, being a covariant realization of the projector operator onto physical states, and we identified explicitly in the model the Z_2 symmetry that characterizes it. We have shown how to break this symmetry consistently to obtain a spin foam realization of a causal transition amplitudes between quantum gravity states. The resulting spin foam model[1] is a path integral for Lorentzian 1st order Regge calculus with an algebraic characterization of all the geometric variables and with a clear definition of the integration measure. It is the first non-trivial example of a causal spin foam model, fits into the framework of quantum causal histories and links several areas of research: canonical and sum-over-histories formulation of quantum gravity, Regge calculus, causal sets and dynamical triangulations.

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